FREQUENCY BASED STOCHASTIC DIFFERENCE EQUATIONS FOR DE-NOISE THE DIGITAL SIGNAL

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ABSTRACT:

This article studied a coupled general nonlinear Frequency based stochastic difference equations (FBSDE). The technique used here is to determine a consistent relationship between the backward and forward pattern that decouples the FBSDEs. The same technique is used as a major contribution to de-noise the digital signal during packet transfer from source to destination node. Optimal controller is established by coupled FBSDE’s and authenticated this through positive definite matrix. The result applied in to the linear quadratic optimal filtering of various digital systems with increasing random noise.

Keywords: Stochastic difference equations, de-noise, digital signal, quadratic optimal filter.
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I. INTRODUCTION

Frequency based stochastic difference equations (FBSDEs) have wide applications in engineering and technology. Once the FBSDEs are decoupled or partially paired, then the specific coupled cases is characterized by the response and establishing a relationship between the forward and backward variables [4]. As a counterpart in the case of discrete time, the solvability of forward and backward random differential equations has been extensively studied by various researches and decoupled in terms of forward and backward coupled random difference equations [1]. Furthermore, the communication engineering problems especially which involves the discrete time random variables are governed by FBSDEs and are of great importance in the theory Digital Signal Processing [5,7]. Few researchers approximates the forward and random difference equations for FBSDEs coupled equations and reverse the same for decoupled constraints [3]. The question of optimality was thought in [12] for the randomized system guided by FBSDEs with partial data [9]. In addition to the broad applications of FBSDEs, motivates the researchers to investigate the solubility of fully coupled FBSDEs, which remains a challenging one in any data analysis [11].

The rest of the article is organized as follows. Section 2 formulates the FBSDEs studied in the paper. The distinct and unique response of the FBSDEs is mentioned and the specified optimality its optimality. Some concluding remarks are provided in section 3.

II. PROBLEM FORMULATION

Consider the following FBSDE,

\[
\begin{align*}
x_{k+1} &= A_k x_k + B_k E_{k-1}(C'_i \lambda_k), \\
\lambda_{k-1} &= E_{k-1}[D_k \lambda_k] + Q x_k, \\
\lambda_{N} &= H x_{N+1}
\end{align*}
\]

where \(x_k \in R^n\) is the forward element and \(\lambda_k \in R^m\) is that the backward element. \(x_i\) be the intermediate of initial and final values. \(A_k, B_k, C_k, C_i, D_k, E_k, F, G, H, N, Q, R, S, T, U, V, W, \lambda_k, w_k\) are constant matrices and \(w_k\) could be a noise with zero mean and variance. \(Q, H, N \in R^{m \times n}\) are constant matrices and \(C_i\) be the conjugate of the matrix \(C\) and the conditional expectation of noise is denoted by \(E_k\).
**Definition 1:** A combination of forward and backward elements \((x_{k+1}, \lambda_{k-1})\) is called the response of FBSDE and the initial and final values are \((i, k)\) which satisfies the following [10]

\[
x_{k+1} = \Phi_1(k, l)x + \sum_{i=1}^{k} \Phi_2(k, i + 1)B_i \times E_{i-1}(C_i \lambda_k),
\]

\[
\lambda_{k-1} = E_{k-1} [\Phi_2(k, N) \lambda_N] + \sum_{i=k-1}^{N} E_{k-1} [\Phi_2(k, i)Q x_{i+1}]
\]

(1)

where \(\Phi_1(k, i) = A_{ik}, \ldots , A_i, i \leq k, \Phi_1(k, i) = I, i > k\) and \(\Phi_2(k, i) = D_{ki}, \ldots , D_i, i \leq k, \Phi_2(k, i) = I, i > k\).

**Theorem 1:** The following statements are equivalent:

The FBSDE (1) have a unique solution.

The following Riccati equation admits the matrix \(I - C'^p k+1 B - \overline{C'} P_{k+1} \overline{B}\) and is invertible:

\[
P_k = D P_{k+1} A + \overline{D} P_{k+1} \overline{A} + (D P_{k+1} B + \overline{D} P_{k+1} \overline{B}) \left( I - C'^p k+1 B - \overline{C'} P_{k+1} \overline{B} \right)^{-1} \times (C'^p k+1 A + \overline{C'} P_{k+1} \overline{A}) + Q
\]

(2)

where \(P_{N+1} = H\), in this case, the distinct solution of FBSDE (1) is given by \((P_k x_k, x_k)\) and \(P_k\) is outlined by (2) and \(x_k\) obeys the following subsequent dynamics [8]:

\[
x_{k+1} = A_k + B_k (I - C'^p k+1 B - \overline{C'} P_{k+1} \overline{B})^{-1} \times (C'^p k+1 A + \overline{C'} P_{k+1} \overline{A}) + Q
\]

(3)

Let \(\gamma_{k+1} = 1 - \begin{bmatrix} BC' P_{k+1} & B \overline{C'} P_{k+1} \\ B \overline{C'} P_{k+1} & B \overline{C'} P_{k+1} \end{bmatrix}\) if the FBSDE (1) have an explicit solution. And so

\[
x_{N+1} = A_N x_N + B_N E_{N-1}(C'_N H x_{N+1})
\]

(4)

Then consider

\[
\left(I - \begin{bmatrix} BC' H & B \overline{C'} H \\ B \overline{C'} H & B \overline{C'} H \end{bmatrix}\right) \begin{bmatrix} E_{N-1}(x_{N+1}) \\ E_{N-1}(w_N x_{N+1}) \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} x_N
\]

(5)

This leads to the explicit solution and let \([\gamma_{N+1}^i, \gamma_{N+1}^{i-1}], i = 1, 2\) be the solutions of (5), and we have [6],

\[
\begin{bmatrix} \gamma_{N+1}^i \\ \gamma_{N+1}^{i-1} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} x_N
\]

(6)

Denote \(z_{N+1}^i = y_{N+1}^i + w_N y_{N+1}^{i-1}, y_{N+1}^i = E_{N-1}(x_{N+1}^i)\) and \(y_{N+1}^{i-1} = E_{N-1}(w_N z_{N+1}^i)\). Then

\[
z_{N+1}^i = A_N x_N + B_N C' H y_{N+1}^i + B \overline{C'} H \gamma_{N+1}^{i-1} + w_N A x_N + w_N B C' H y_{N+1}^i + w_N B \overline{C'} H \gamma_{N+1}^{i-1}
\]

\[
= A_N x_N + B_N C' H y_{N+1}^i + B \gamma_{N+1}^{i-1} + B \overline{C'} H \gamma_{N+1}^{i-1}
\]

\[
= A_N x_N + B_N E_{N-1}(C'_N H z_{N+1}^i)
\]

This implies that \(z_{N+1}^i (i = 1, 2)\) and shows that solution exist for (4). Observe that \(E \|z_{N+1}^1 - z_{N+1}^2\|^2 = E \|y_{N+1}^1 - y_{N+1}^2\|^2 + E \|y_{N+1}^{i-1} - y_{N+1}^{i-2}\|^2\). Combining (4), with this, we have \(y_{N+1}^1 = y_{N+1}^2\) and \(y_{N+1}^{i-1} = y_{N+1}^{i-2}\). Consequently, there is a unique solution for (5). This results in the invertibility of the matrices (5), and further [2]

\[
\begin{bmatrix} E_{N-1}(x_{N+1}) \\ E_{N-1}(w_N x_{N+1}) \end{bmatrix} = \gamma_{N+1}^{i-1} \begin{bmatrix} A \end{bmatrix} x_N
\]

Together with the second equation of (1), it follows that \(\lambda_{N+1} = P_N x_N\) where
\[ P_N = [DP_{N+1} \; DP_{N+1}] \gamma_{N+1}^{-1} \begin{bmatrix} A \\ \bar{A} \end{bmatrix} + Q \]

Assuming that with \( \lambda_k = P_{k+1} x_{k+1} \), where \( P_s = [DP_{s+1} \; DP_{s+1}] \gamma_{s+1}^{-1} \begin{bmatrix} A \\ \bar{A} \end{bmatrix} + Q \) for \( k + 1 \leq s \leq N \) with \( P_{N+1} = H \), and execute this for the time span \( k \). Applying similar arguments to time \( N \), we have the distinct solutions of the equation.

\[ \gamma_{k+1} \begin{bmatrix} E_{k-1}(x_{k+1}) \\ E_{k-1}(w_kx_{k+1}) \end{bmatrix} = \begin{bmatrix} A \\ \bar{A} \end{bmatrix} x_k \]  

(7)

This produces that

\[ \begin{bmatrix} E_{k-1}(x_{k+1}) \\ E_{k-1}(w_kx_{k+1}) \end{bmatrix} = \gamma_{k+1}^{-1} \begin{bmatrix} A \\ \bar{A} \end{bmatrix} x_k \]

Combining with the second equation of FBSDE (1), we have

\[ \lambda_{k-1} = P_k x_k, \]

Where \( P_k \) satisfies the following Riccati equation:

\[ P_k = [DP_{k+1} \; DP_{k+1}] \gamma_{k+1}^{-1} \begin{bmatrix} A \\ \bar{A} \end{bmatrix} + Q \]

(8)

With \( P_{N+1} = H \). Victimization this formula for the invertibility of the following matrix,

\[ \gamma_{k+1}^{-1} = \left( I - \begin{bmatrix} B C' P_{k+1} & B C' P_{k+1} \\ B C' P_{k+1} & B C' P_{k+1} \end{bmatrix} \right)^{-1} \]

\[ = \left( I - \begin{bmatrix} B \\ \bar{B} \end{bmatrix} \begin{bmatrix} C & \bar{C} & P_{k+1} & 0 & 0 \\ P_{k+1} & C & \bar{C} & P_{k+1} \end{bmatrix} \right)^{-1} \]

\[ = I - \begin{bmatrix} B \\ \bar{B} \end{bmatrix} \left( I - \begin{bmatrix} C' P_{k+1} B & \bar{C} P_{k+1} \bar{B} \end{bmatrix} \right)^{-1} \times \begin{bmatrix} P_{k+1} & 0 & 0 \\ 0 & P_{k+1} & 0 \end{bmatrix} \]

(9)

Hence (2) follows from (8). The specified result is obtained by the induction technique.

It is appropriate to indicate that \( (\lambda_{k-1}, x_k) = (P_k x_k, x_k) \) solves (1) where \( P_k \) is described in (2) and \( x_k \) satisfies (3). Indeed,

\[ \begin{bmatrix} E_{k-1}(x_{k+1}) \\ E_{k-1}(w_kx_{k+1}) \end{bmatrix} = \gamma_{k+1}^{-1} \begin{bmatrix} A \\ \bar{A} \end{bmatrix} x_k \]

Combining the above equation with (8), it produces that

\[ P_k x_k = [DP_{k+1} \; DP_{k+1}] \gamma_{k+1}^{-1} \begin{bmatrix} A \\ \bar{A} \end{bmatrix} x_k + Q x_k \]

\[ = [DP_{k+1} \; DP_{k+1}] \begin{bmatrix} E_{k-1}(x_{k+1}) \\ E_{k-1}(w_kx_{k+1}) \end{bmatrix} + Q x_k \]

(10)

While comparing (3), (10) and (1), together we have got that \( (P_k x_k, x_k) \) could be a resolution for FBSDE (1) leads to the uniqueness. Let \( \zeta_k = \lambda_k - P_{k+1} x_{k+1} \) and consider the following FBSDEs:

\[ x_{k+1} = A_k x_k + B_k E_{k-1}(C'_i P_{k+1} x_{k+1} + C'_i \zeta_k), \]

\[ \zeta_{k-1} = E_{k-1}(D_k \zeta_k) [DP_{k+1} \; DP_{k+1}] \gamma_{k+1}^{-1} \begin{bmatrix} B \\ \bar{B} \end{bmatrix} E_{k-1}(G_k \zeta_k) \]

(11)

(12)
Observing that $\zeta_N = \lambda_N - P_{N+1}x_{N+1} = 0$, and $\zeta_{k-1} = 0$, $k \geq 0$ leads the uniqueness of (12). In this case, (11) reduces to $x_{k+1} = A_k x_k + B_k E_{k-1}(C_i P_{k+1} x_{k+1})$ Victimizing the invertibility of the matrix $I - C_i P_{k+1} B - C_i P_{k+1} B$, we tend to follow the distinct response of $x_k$. This means that $(P_k x_k, x_k)$ is the only solution for FBSDEs (1).

Now extend FBSDE (1) with the parameters of the particular matrix $C_k = -R^{-1}B_k$, $D_k = A_k^t$ where $R \in R_{nxn}$ is positive definite matrix, and can be written like

$$x_{k+1} = A_k x_k + B_k R^{-1}E_{k-1}(B_k^t \lambda_k), \quad x_0 = x$$

$$\lambda_{k-1} = E_{k-1} [A_k^t \lambda_k] + Q x_k, \quad \lambda_N = H x_{N+1}$$

(13)

In (7), if $m = n$ then we can say that $Q, H$ are positive semi-definite matrices. Now introduce the Riccati difference equation in correlation with (13) as

$$P_k = A'P_{k+1}A + \bar{A}'P_{k+1}\bar{A} - (A'P_{k+1} B + \bar{A}'P_{k+1}\bar{B})(R + B'P_{k+1} B + B'P_{k+1}\bar{B})^{-1} \times (B'P_{k+1} A + B'P_{k+1}\bar{A}) + Q$$

(14)

**Corollary 1:** The following are equivalent of subsequent declarations:

FBSDE (13) has a new solution;

Riccati’s equation (14) admits a finite solution with $R + B'P_{k+1} B + \bar{B}'P_{k+1}\bar{B} > 0$

In this case, the one and only solution of FBSDE (13) is given by $(\lambda_{k-1}, x_k) = (P_k x_k, x_k)$ where $P_k$ satisfies (14) and $x_k$ obeys the following

$$x_{k+1} = A_k - B_k (R + B'P_{k+1} B + \bar{B}'P_{k+1}\bar{B})^{-1} \times (B'P_{k+1} A + B'P_{k+1}\bar{A})] x_k, \quad x_0 = x$$

(15)

**Proof:** Consider the FBSDE (13) and the fact that the Riccati equation’s solution satisfies that $I + R^{-1}B'P_{k+1} B + R^{-1} \bar{B}'P_{k+1}\bar{B}$. Furthermore

$$P_k = A'P_{k+1} A + \bar{A}'P_{k+1}\bar{A} - (A'P_{k+1} B + \bar{A}'P_{k+1}\bar{B})(I + R^{-1} B'P_{k+1} B + R^{-1} \bar{B}'P_{k+1}\bar{B})^{-1} \times (R^{-1}B'P_{k+1} A + R^{-1} \bar{B}'P_{k+1}\bar{A}) + Q$$

Formulating the previous equation, we have the Riccati equation (14). Taking into account that $R > 0$ and $P_{k+1} \geq 0$ [1], we have $R + B'P_{k+1} B + \bar{B}'P_{k+1}\bar{B} > 0$. This means that Riccati’s equation (14) is often solved. The proof is now complete.

Consider the posterior system of FBSDE (13)

$$x_{k+1} = A_k x_k + B_k u_k, \quad x_1 = x$$

(16)

This executes

$$x_{k+1}(l, x; u) = E[\sum_{k=1}^{N} (x'_k Q x_k + u'_k R u_k) + x'_{N+1} H x_{N+1}]$$

(17)

and

$$u_k = -(R + B'P_{k+1} B + \bar{B}'P_{k+1}\bar{B})^{-1}(B'P_{k+1} A + \bar{B}'P_{k+1}\bar{A})$$

Where $Q \geq 0, H \geq 0, R > 0$.

The optimal controller of $u_k$ could written like

$$u_k = -R^{-1}E_{k-1}[B_k^t \lambda_k]$$

These are the equivalent conditions to the specified optimization problems.

**III. CONCLUSION**

This paper examined a category of coupled general linear FBSDEs. In continuation, the persistence relationship between the backward and forward pattern that decouples the FBSDEs are arrived through Riccati difference equations. This technique is applied in de-noising the communication signal. Optimal system controller is
applied in reverse basis like applying the coupled FBSDE's and authenticated this through positive definite matrix
with increasing random noise.

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