TOTAL EDGE DETOUR DOMINATION NUMBER OF GRAPHS

Dr. A. MAHALAKSHMI
Department of Mathematics, Sri Sarada College for Women (Autonomous),
Tirunelveli-627011, Tamilnadu,
India, lakshmi19maths@gmail.com

ABSTRACT

In this paper, total edge detour domination number of a graph is defined and is found for some standard graphs. Also, bounds are obtained.

Keywords: Detour, Domination, Detour domination, edge detour domination.

1. INTRODUCTION

The concept of domination was introduced by Ore and Berge [8]. Let G be a finite, undirected connected graph with neither loops nor multiple edges. A subset D of V(G) is a dominating set of G if every vertex in V-D is adjacent to at least one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number γ(G) of G. We consider connected graphs with at least two vertices. For basic definitions and terminologies, we refer Harary [3]. For vertices u and v in a connected graph G, the detour distance D(u,v) is the length of longest u-v path in G. A u-v path of length D(u,v) is called a u - v detour. A subset S of V is called a detour set if every vertex in G lie on a detour joining a pair of vertices of S. The detour number dn(G) of G is the minimum order of a detour set and any detour set of order dn(G) is called a detour basis of G. These concepts were studied by Chartrand [4]. A subset S of V is called an edge detour set of G if every edge in G lie on a detour joining a pair of vertices of S. The edge detour number d\textsubscript{ed}(G) of G is the minimum order of its edge detour sets and any edge detour set of order d\textsubscript{ed} is an edge detour basis. A graph G is called an edge detour graph if it has an edge detour set. Edge detour graphs were introduced and studied by Santhakumaran and Athisayanathan [10]. Let G be a connected graph with at least two vertices. An edge detour dominating set is a subset S of V(G) which is both a dominating and an edge detour set of G. An edge detour dominating set is said to be minimal edge detour dominating set if there exists no edge detour dominating set S’ such that S’ is the smallest cardinality of an edge detour dominating set of G is called the edge detour domination number of G. It is denoted by (γ, eD). Any edge detour dominating set G of minimum cardinality is called γ\textsubscript{eD}-set of G. The edge detour domination number of graphs were introduced and studied by A.Mahalakshmi,K.Palani and S.Somasundaram [7].

Theorem 1.1 ([6]). The domination number of some standard graphs are given as follows.

1. \(γ(P_p) = \lceil \frac{p}{3} \rceil, p \geq 3.\)
2. \(γ(C_p) = \lceil \frac{p}{3} \rceil, p > 3.\)
3. \(γ(K_p) = γ(W_p) = γ(K_{1,n}) = 1.\)
4. \(γ(K_{m,n}) = 2\) if \(m, n \geq 2.\)

Theorem 1.2. Every end vertex of an edge detour graph G belongs to every edge detour set of G. Also, if the set of all end vertices of G is an edge detour set, then S is the unique edge detour basis for G.
Theorem 1.3.

\[ \gamma_{eD}(P_n) = \begin{cases} \left\lceil \frac{n - 4}{3} \right\rceil + 2 & \text{if } n \geq 5 \\ 2 & \text{if } n = 2, 3 \text{ or } 4. \end{cases} \]

Remark 1.4. Let \( G \) be a connected graph with \( p \geq 2 \) vertices. Then,

(i) \( \gamma(G) \leq \gamma_{eD}(G) \).

(ii) \( d_n(G) \leq \gamma_{eD}(G) \).

(iii) Strict inequality is also true in the above relations. For example,

a) Consider \( P_{11} \). Let \( V(P_{11}) = \{ v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11} \} \).

\[ S_1 = \{ v_1, v_4, v_7, v_9, v_{11} \} \text{ is a } \gamma_{eD}-\text{set of } P_{11} \text{ and so } \gamma_{eD}(P_{11}) = 5. \]

But, \( \gamma(P_{11}) = \left\lceil \frac{11}{3} \right\rceil = 4 < \gamma_{eD}(P_{11}) \).

b) Consider, \( P_8 \). \( d_n(P_8) \neq 2 \) and \( \gamma_{eD}(P_8) = 4 \). Therefore, \( d_n(P_8) < \gamma_{eD}(P_8) \).

I. TOTAL EDGE DETOUR DOMINATION NUMBER OF GRAPHS

Definition 2.1: Let \( G \) be a connected graph. An edge detour dominating set \( S \) of \( G \) is called a total edge detour dominating set of \( G \) if the subgraph induced by \( S \) has no isolated vertex. The total edge detour domination number \( T_{\gamma_{eD}}(G) \) of \( G \) is the minimum cardinality of its total edge detour dominating sets and any total edge detour dominating set of cardinality \( T_{\gamma_{eD}}(G) \) is called a \( T_{\gamma_{eD}} \)-set of \( G \).

Example 2.2. Consider the graph \( G \) in figure 1.

Here, \( S = \{ v_1, v_2, v_5, v_6 \} \) is the unique minimum edge detour dominating set of \( G \) and so \( \gamma_{eD}(G) = 4 \).

Also, the subgraph induced by \( S \) has no isolated vertices. Therefore, is a total edge detour dominating set of \( G \).

Hence, \( T_{\gamma_{eD}}(G) = 4 \).

Example 2.3. Consider the graph \( G \) in figure 2.
Figure 2

Here, \( S = \{v_1, v_4\} \) is the unique minimum edge detour dominating set of \( G \) and so \( \gamma_{ed}(G) = 2 \). But, the subgraph induced by \( S \) has two isolated vertices. Therefore, \( S \) is not a total edge detour dominating set of \( G \).

Also, \( S_1 = \{v_1, v_2, v_3, v_4\} \) and \( S_2 = \{v_1, v_2, v_4, v_5\} \) are the total edge detour dominating sets of \( G \). Hence, \( T\gamma_{ed}(G) = 4 \).

**Theorem 2.4.** Every end vertex of \( G \) belongs to every total edge detour dominating set of \( G \).

**Proof.** Since every total edge detour dominating set is an edge detour dominating set of \( G \), the result follows from Remark 1.4.

**Theorem 2.5.** Let \( G \) be a connected graph and \( uv \) be a pendant edge of \( G \). Then \( \{u, v\} \) is a subset of every total edge detour dominating set of \( G \).

**Proof.** Let \( G \) be a connected graph. Let \( uv \) be a pendant edge of \( G \) with \( v \) as the end vertex. Let \( S \) be a total edge detour dominating set of \( G \). By 2.4, \( v \in S \).

Suppose \( u \notin S \), then \( v \) is an isolated vertex in the subgraph induced by \( S \), which is a contradiction to \( S \) is a total edge detour dominating set of \( G \). Therefore, \( u \in S \).

Hence, \( \{u, v\} \) is a subset of every total edge detour dominating set of \( G \).

**Theorem 2.6.** For a graph \( G \) of order \( p \geq 2 \), \( 2 \leq \gamma_{ed}(G) \leq \gamma_{ed}(G) \leq p \).

**Proof.** Always, \( \gamma_{ed}(G) \geq 2 \).

Every total edge detour dominating set is an edge detour dominating set.

Therefore, \( \gamma_{ed}(G) \leq T\gamma_{ed}(G) \).

Clearly, \( T\gamma_{ed}(G) \leq p \).

Hence, \( 2 \leq \gamma_{ed}(G) \leq T\gamma_{ed}(G) \leq p \).

**Remark 2.7.** The bounds in previous Theorem are strict.

For, the graph \( G \) given in figure 3.

![Figure 3](image-url)

Here, \( p = 6 \), \( S = \{v_1, v_3, v_6\} \) is an edge detour dominating set of \( G \) and so \( \gamma_{ed}(G) = 3 \). But, the subgraph induced by \( S \) has three isolated vertices. Therefore, \( S \) is not a total edge detour dominating set of \( G \). And also, \( S_1 = \{v_1, v_2, v_5, v_6\} \); \( S_2 = \{v_1, v_2, v_4, v_6\} \) are the total edge detour dominating sets of \( G \) and so \( T\gamma_{ed}(G) = 4 \).

Hence, \( 2 < \gamma_{ed}(G) < T\gamma_{ed}(G) < p \).
Observation 2.8.
1. $T_{\gamma_{eD}}(K_n) = 3$.
2. $T_{\gamma_{eD}}(K_{1,n}) = n + 1$.
3. $T_{\gamma_{eD}}(P_2) = 2$
4. $T_{\gamma_{eD}}(P_3) = 3$

Theorem 2.9.

For $k \geq 1$, $T_{\gamma_{eD}}(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 2 & \text{if } n = 4k \\ \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } n = 4k + 2 \text{ or } 4k + 3 \end{cases}$.

Proof. Case (i): Let $n = 4k$.

Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_{4k-2}, v_{4k-1}, v_{4k}\}$. Choose the first two of every four vertices of $V(P_n)$ starting from $v_1$ to get a minimum total edge detour dominating set of $P_n$.

And so, we get $\{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k-3}, v_{4k-2}\}$.

Since, $v_{4k}$ is an end vertex of $P_n$, every total edge detour dominating set contains $\{v_{4k-1}, v_{4k}\}$.

Hence, $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k-3}, v_{4k-2}\} \cup \{v_{4k-1}, v_{4k}\}$ is a minimum total edge detour dominating set of $P_n$.

Therefore, $T_{\gamma_{eD}}(P_n) = |S|$

$= 2k + 2$

$= \left\lceil \frac{n}{2} \right\rceil + 2$ where $n = 4k$.

Case (ii): $n = 4k + 1$.

Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_{4k-2}, v_{4k-1}, v_{4k}, v_{4k+1}\}$.

Proceeding as before, to get a minimum total edge detour dominating set of $P_n$, choose the first two of every four vertices of $V(P_n)$ starting from $v_1$ and so we get $\{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k-3}, v_{4k-2}\}$.

Since, $v_{4k+1}$ is an end vertex of $P_n$, every total edge detour dominating set contains $\{v_{4k}, v_{4k+1}\}$.

Therefore, $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k-3}, v_{4k-2}\} \cup \{v_{4k}, v_{4k+1}\}$ is a minimum total edge detour dominating set of $P_n$.

Hence, $T_{\gamma_{eD}}(P_n) = |S|$

$= 2k + 2$

$= (2k + 1) + 1$

Case (iii): $n = 4k + 2$.

Let $V(P_n) = \{v_1, v_2, v_3, \ldots, v_{4k-1}, v_{4k}, v_{4k+1}, v_{4k+2}\}$.

Proceeding as before, $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k-3}, v_{4k-2}\} \cup \{v_{4k+1}, v_{4k+2}\}$ is a minimum total edge detour dominating set of $P_n$.

Hence, $T_{\gamma_{eD}}(P_n) = |S|$

$= 2k + 2$

$= (2k + 1) + 1$
\[= \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+2.\]

Case (iv): \(n = 4k + 3.\)

Let \(V(P_n) = \{v_1, v_2, v_3, \ldots, v_{4k+2}, v_{4k+3}, v_{4k+4}\}.\) Proceed as before starting from \(v_1,\) till we reach \(v_{4k}.\)

Hence, \(\{v_1, v_2, v_5, v_6, \ldots, v_{4k-3}, v_{4k-2}\}\) is a subset of every total edge detour dominating set of \(P_n.\)

Now, the left out vertices are \(\{v_{4k+1}, v_{4k+2}, v_{4k+3}\}.\)

Here, \(v_{4k+3}\) is an end vertex. Therefore, \(\{v_{4k+2}, v_{4k+3}\}\) dominates the vertex \(v_{4k}.\)

Further, \(S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k-3}, v_{4k-2}\}\) is not dominating the vertex \(v_{4k}.\)

Hence, \(T_{\gamma eD}(P_n) = |S| + 1 = \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+3.\)

\[= \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+2.\]

\[= \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+3.\]

\[= \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+2.\]

\[= \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+3.\]

\[= \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+2.\]

\[= \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+3.\]

\[= \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+2.\]

\[= \left\lceil \frac{n}{2} \right\rceil + 1 \text{ where } n = 4k+3.\]

Theorem 2.10. \(T_{\gamma eD}(C_4) = T_{\gamma eD}(C_5) = 3; T_{\gamma eD}(C_6) = T_{\gamma eD}(C_7) = 4\) and for \(k > 1.\)

\[T_{\gamma eD}(C_n) = \begin{cases} 
\left\lceil \frac{n}{2} \right\rceil & \text{if } n = 4k \text{ or } 4k + 1 \\
\left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } n = 4k + 2 \text{ or } 4k + 3.
\end{cases}\]

Proof.

Let \(V(C_n) = \{v_1, v_2, \ldots, v_n\}.\) Here, \(S = \{v_1, v_2, v_3\}\) is the minimum total edge detour dominating set of \(C_n.\)

Hence, \(T_{\gamma eD}(C_n) = 3.\)

When \(n = 6\) or \(7, S = \{v_1, v_2, v_4, v_5\}\) is one of the minimum total edge detour dominating set of \(C_n.\)

Hence, \(T_{\gamma eD}(C_6) = 4 = T_{\gamma eD}(C_7).\)

Let \(k > 1.\)

Case (i): \(n = 4k.\)

Let \(V(C_n) = \{v_1, v_2, v_3, \ldots, v_{4k-1}, v_{4k}\}.\)

Now, choose the first two of every four vertices of \(V(C_n)\) starting from \(v_1\) to get a minimum total edge detour dominating set.

Hence, \(S = \{v_1, v_2, v_5, v_6, v_{10}, \ldots, v_{4k-3}, v_{4k-2}\}\) is a minimum total edge detour dominating set of \(C_n.\)

Therefore, \(T_{\gamma eD}(C_n) = 2k = \frac{n}{2} = \left\lceil \frac{n}{2} \right\rceil\) when \(n = 4k.\)

Case (ii): \(n = 4k + 1.\)

Let \(V(C_n) = \{v_1, v_2, v_3, \ldots, v_{4k}, v_{4k+1}\}.\)

Proceeding as before, \(S = \{v_1, v_2, v_5, v_6, \ldots, v_{4k-3}, v_{4k-2}\}\) is a subset of every total edge detour dominating set of \(C_n.\)

But, \(S\) neither dominates all the vertices of \(C_n\) (\(S\) does not dominate \(v_{4k}\)) nor it is an edge detour set.

But, \(S \cup \{v_{4k-1}\}\) and \(S \cup \{v_{4k+1}\}\) are total edge detour dominating sets of \(C_n\) with minimum cardinality.

Hence, \(T_{\gamma eD}(C_n) = |S| + 1\)
\[ = 2k + 3 \]
\[ = (2k + 2) + 1 \]
\[ = \left\lceil \frac{n}{2} \right\rceil \] when \( n = 4k + 1 \).

**Theorem 2.11.** For the complete bipartite graph, total edge detour domination number is 3.

*Proof.* Let \( V(K_{m,n}) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n\} \) and \( |V_1| = m; \]
\[ |V_2| = n. \]
Any three element set containing two vertices from \( V_1 \) and one vertex from \( V_2 \) and vice versa forms a minimum total edge detour dominating set.

Therefore, \( T_{\gamma eD}(K_{m,n}) = 3 \).

**Theorem 2.12.** If \( G \) is a caterpillar with \( p \) vertices such that for every cut vertex \( v \) of \( G \) there exists a pendant vertex, then \( T_{\gamma eD}(G) = p \).

*Proof.* Since for every cut vertex \( v \) of \( G \), there exists a vertex \( u \) such that \( uv \) is a pendant edge of \( G \) by 2.4 and 2.5, \( \{u, v\} \) is a subset of every total edge detour dominating set of \( G \) for every \( v \in G \). Hence, it follows that \( T_{\gamma eD}(G) = p \).

**Theorem 2.13.** For the wheel graph \( W_{1,p-1} \), \( T_{\gamma eD}(W_{1,p-1}) = 3 \).

*Proof.* Let \( G = W_{1,p-1} \) and \( V(W_{1,p-1}) = \{v, v_1, v_2, v_3, \ldots, v_{p-1}\} \).
Here, \( S = \{v, v_1, v_2\} \) forms an edge detour dominating set of \( W_{1,p-1} \) and \( |S| = 3 \). Also, the subgraph induced by \( S \) has no isolated vertex. Further, no two element set \( \{u, v\} \) containing adjacent vertices is an edge detour set, since the edge \( uv \) does not lie in any edge detour joining the vertices \( u \) and \( v \).
Therefore, \( S \) is the minimum total edge detour dominating set of \( G \). Hence, \( T_{\gamma eD}(W_{1,p-1}) = 3 \).

**Problem 2.14.** For any two integers \( a, b \) and \( p \) with \( 2 \leq a \leq b \leq 2a \), there exists a connected graph \( G \) of order \( p \) with \( \gamma eD(G) = a \) and \( T_{\gamma eD}(G) = b \).

**Remark 2.15.** \( 2 \leq \gamma eD(G) \leq T_{\gamma eD}(G) \leq 2\gamma eD(G) \).

Here, the bounds are sharp. For, \( 2 = a = b \), \( P_2 \) satisfies the condition. For, \( 2 = a < b = 2a \), \( P_4 \) satisfies the condition.

**REFERENCES**